

INSTANTONS IN $\mathcal{N} = 2$ $Sp(N)$ SUPERCONFORMAL GAUGE THEORIES AND THE ADS/CFT CORRESPONDENCE

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Abstract

We study, using ADHM construction, instanton effects in an $\mathcal{N} = 2$ superconformal $Sp(N)$ gauge theory, arising as effective field theory on a system of N D-3-branes near an orientifold 7-plane and 8 D-7-branes in type I' string theory. We work out the measure for the collective coordinates of multi-instantons in the gauge theory and compare with the measure for the collective coordinates of (-1) -branes in the presence of 3- and 7-branes in type I' theory. We analyse the large- N limit of the measure and find that it admits two classes of saddle points: In the first class the space of collective coordinates has the geometry of $AdS_5 \times S^3$ which on the string theory side has the interpretation of the D-instantons being stuck on the 7-branes and therefore the resulting moduli space being $AdS_5 \times S^3$, In the second class the geometry is $AdS_5 \times S^5/Z_2$ and on the string theory side it means that the D-instantons are free to move in the 10-dimensional bulk. We discuss in detail a correlator of four $O(8)$ flavour currents on the Yang-Mills side, which receives contributions from the first type of saddle points only, and show that it matches with the correlator obtained from F^4 coupling on the string theory side, which receives contribution from D-instantons, in perfect accord with the AdS/CFT correspondence. In particular we observe that the sectors with odd number of instantons give contribution to an $O(8)$ -odd invariant coupling, thereby breaking $O(8)$ down to $SO(8)$ in type I' string theory. We finally discuss correlators related to R^4 , which receive contributions from both saddle points.

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1 Introduction

Dualities, and in particular strong-weak coupling dualities, have shed light on our understanding of the dynamics of string theories and Yang-mills theories. More recently the conjecture of Maldacena [1] relates the physics of certain conformally invariant large N Yang-Mills theories living on 3-branes in various string theories to that of the corresponding bulk string theories in the near horizon geometry (AdS geometry) of the 3-branes. Many detailed checks have so far confirmed this conjecture and have at the same time provided new insights into both the physics of Yang-Mills theories as well as that of string theories. In a beautiful paper [2], in the context of $\mathcal{N} = 4$ Yang-Mills theories, this equivalence was extended to e^{-N} orders. On the Yang-Mills side they appear as instanton effects, while in the string theory they correspond to certain higher derivative couplings induced by stringy instantons. What was remarkable was that in the large N limit, the moduli space of $SU(N)$ multi-instantons collapses into the "center of mass" moduli that live on $AdS_5 \times S^5$ as predicted by AdS/ CFT correspondence, and a set of "relative" moduli whose action is that of a multi D-instanton action in the

underlying IIB string theory. It was known through the works [3, 4] and [5] that the latter contributes to certain R^4 terms in the IIB theory (R being the Riemann tensor). Also, it was shown that certain correlators in the Yang-Mills instanton background, had a direct correspondence with certain couplings appearing in the string theory via the dictionary of the Yang-Mills operators and the bulk operators as given in [6, 7] .

The purpose of this paper is to extend the results of ref. [2] to the case of $\mathcal{N} = 2$ $Sp(N)$ Yang-Mills theories that appear as the 3-brane world volume theory in the type I' model where the 3-branes are living at an orientifold 7-plane together with 8 D-7-branes. The near horizon geometry of this system is that of $AdS_5 \times X_5$ where X_5 is a particular Z_2 modding of S^5 . The 7-brane world volume intersects with X_5 on an S^3 . This model was first studied in refs. [8, 9] following the earlier works [10, 11]. Subsequently the order N corrections to the AdS/CFT trace anomaly in this model were analyzed in refs. [12, 13]. In the bulk theory this correction was traced, in ref. [13], to the presence of an R^2 term in the 7-brane world volume action.

In the present work we will consider instantons in the 3-brane world volume theory and show that in the large N limit, two types of saddle points are relevant: Those for which multi-instanton moduli space is $AdS_5 \times S^3$, and the fluctuations are given by the D-instanton action in type I' theory, and those for which the multi-instanton moduli space is $AdS_5 \times S^5/Z_2$, and the fluctuations are given by the D-instanton action in type IIB theory. For the former saddle points the interpretation on the string side is that the D-instantons sit at a point on the 7-branes whereas for the latter ones they can be at a point in the whole 10 dimensional bulk. We will study certain correlation functions, in the Yang-Mills theory, of the $O(8)$ flavour currents that couple to $O(8)$ gauge fields in the bulk via AdS/ CFT correspondence. We will see that in the large N limit the first type of saddle points are relevant for these correlation functions. Moreover, we will show that these correlation functions describe the bulk propagators that connect F^4 vertex to the $O(8)$ currents on the boundary. The result then is obtained by the D-instanton contributions to F^4 in type I' theory or equivalently D-string instanton contributions in type I theory [14, 15] or world sheet instantons at string one-loop level in heterotic theory [16, 17]. We will see that both odd and even instanton numbers contribute to this correlator. We will also see that certain correlators can receive contributions from instanton configurations whose moduli space is the full 10-dimensional bulk spacetime $AdS_5 \times S^5/Z_2$.

Some of the issues developed here have been recently discussed in [18]. Also instanton effects in $SU(N)$ superconformal $\mathcal{N} = 2$ gauge theories have been discussed in [19] and, in the context of orbifold AdS/CFT correspondence, in [20]

The paper is organized as follows. In section 2 we review the 3-brane world volume theory and describe the relevant ADHM construction of the multi-instantons. In section 3 we describe the ADHM measure in terms of integrals over the fields on (-1) -branes in a system of (-1) -, 3- and 7-branes. We then go on in section 4 to find the saddle

point solutions in the large- N limit and show that the integral collapses to a "center of mass" integral on $AdS_5 \times S^3$ (i.e. in the 7-brane world volume where $O(8)$ gauge fields live) or $AdS_5 \times S^5/Z_2$, together with integrals over the fluctuations describing the relative positions of D-instantons in type I' or IIB theory respectively. For the first saddle point, we will see that both odd and even instantons contribute to the correlation function involving four $O(8)$ currents and give rise to odd and even quartic invariants of $O(8)$. In section 5 we show that the four $O(8)$ current correlators describe bulk to boundary propagators connecting F^4 vertex in AdS_5 to four $O(8)$ currents on the boundary of AdS_5 . We will explicitly show the emergence of the usual t_8 tensor in F^4 vertex. In section 6, we describe the heterotic string computation of F^4 term for the odd invariant case since this has not appeared in the literature so far. We will also discuss R^4 couplings, which receive contributions from both saddle points. Finally in section 7 we make concluding remarks.

2 ADHM construction of $\mathcal{N} = 2$ $Sp(N)$ instantons

The theory on the type I' 3-branes that we are considering here is $\mathcal{N} = 2$ $Sp(N)$ Yang Mills theory with one hypermultiplet transforming under the antisymmetric representation of $Sp(N)$ and a fundamental hyper multiplet transforming as $(N, 8)$ under $Sp(N) \times O(8)$ with $O(8)$ being the flavour symmetry arising from the 7-branes. The following table includes the fields on the 3-brane world volume together with their $SO(4)_I \times SO(4)_E \times SO(2)$ transformation properties. Here the subscript I refers to the $SO(4)$ of the world volume (writing $SO(4) = SU(2)^2$, we shall label the quantum numbers by $\alpha, \dot{\alpha}$), E refers to the external $SO(4)$ which is part of the 7-brane world volume (A and Y label the analogous quantum numbers) and finally $SO(2)$ acts on the space transverse to the 7-branes.

D3-brane World-volume content

Fields	$SO(4)_I$	$SO(4)_E$	$SO(2)$	$Sp(N)$	$O(8)$
v_μ	(2, 2)	(1, 1)	0	$N(2N + 1)$	1
φ^\pm	(1, 1)	(1, 1)	± 2	$N(2N + 1)$	1
φ^{AY}	(1, 1)	(2, 2)	0	$N(2N - 1)$	1
λ_α^A	(2, 1)	(2, 1)	+1	$N(2N + 1)$	1
$\bar{\lambda}_{\dot{\alpha}}^A$	(1, 2)	(2, 1)	-1	$N(2N + 1)$	1
λ_α^Y	(2, 1)	(1, 2)	-1	$N(2N - 1)$	1
$\bar{\lambda}_{\dot{\alpha}}^Y$	(1, 2)	(1, 2)	+1	$N(2N - 1)$	1
q^A	(1, 1)	(2, 1)	0	$2N$	8
η_α	(2, 1)	(1, 1)	-1	$2N$	8
$\bar{\eta}_{\dot{\alpha}}$	(1, 2)	(1, 1)	+1	$2N$	8

It is instructive to compare the above fields with those of $2N$ 3-branes in type IIB. There the theory is $\mathcal{N} = 4$ $U(2N)$ gauge theory. Type I' $Sp(N)$ theory is obtained from the $\mathcal{N} = 4$ $U(2N)$ theory by $\Omega \cdot Z_2$ projection (apart from the $O(8)$ fundamental fields which appear from 3-brane-7-brane states and we shall discuss them separately in the following). Recall that the $\mathcal{N} = 4$ theory has an $SU(4)$ R-symmetry group which can be decomposed in terms of $SO(4)_E \times SO(2) \equiv SU(2)_A \times SU(2)_Y \times SO(2)$ that appears in the $\mathcal{N} = 2$ theory. Writing also $SO(4)_I$ as $SU(2)_\alpha \times SU(2)_{\dot{\alpha}}$ we can write the $\mathcal{N} = 4$ fields (all in the adjoint of $U(2N)$) in terms of their representations under $SU(2)_\alpha \times SU(2)_{\dot{\alpha}} \times SU(2)_A \times SU(2)_Y \times SO(2)$ as

$$\begin{aligned} \text{Vectors} & : (2, 2, 1, 1, 0) \\ \text{Fermions} & : (2, 1, 2, 1, +1), (1, 2, 1, 2, +1), (2, 1, 1, 2, -1), (1, 2, 2, 1, -1) \\ \text{Scalars} & : (1, 1, 2, 2, 0), (1, 1, 1, 1, \pm 2) \end{aligned}$$

The last entry here denotes the charge under $SO(2)$.

$\Omega \cdot Z_2$ projection acts as follows: Z_2 is the center of $SU(2)_Y$, while Ω assigns a plus sign to the $Sp(N)$ subalgebra of $U(2N)$ adjoint and assigns minus to the remaining generators of $U(2N)$ namely the ones transforming as the second rank anti-symmetric representation of $Sp(N)$. From this it follows that vectors are $Sp(N)$ adjoint and transform as $(2, 2, 1, 1, 0)$, while the scalars split into two classes: φ^{AY} that transform as anti-symmetric representation of $Sp(N)$ and φ^\pm transforming as adjoint. Similarly fermions transforming as $(2, 1, 2, 1, +1)$ and their complex conjugate $(1, 2, 2, 1, -1)$ are in the adjoint of $Sp(N)$ while the ones transforming as $(1, 2, 1, 2, +1)$ and their complex conjugate $(2, 1, 1, 2, -1)$ are in the antisymmetric representation. This is exactly the field content (apart from the fundamentals of $O(8)$) described above and in fact even their action is just obtained from the $\mathcal{N} = 4$ action by this projection. This fact will be important for us in the following because our strategy will be to use the results of [2] for the case of $U(2N)$ and project by $\Omega \cdot Z_2$. Not only the zero mode analysis of [2] can be adapted for our case but even the action integrals can be carried over from the $\mathcal{N} = 4$ analysis.

2.1 Vector zero modes

In the $U(2N)$ theory the ADHM data for k instanton is given in terms of $(2N + 2k) \times 2k$ matrix Δ of the form ¹

$$\Delta_{\lambda i \dot{\alpha}} = a_{\lambda i \dot{\alpha}} + b_{\lambda i}^\alpha x_{\alpha \dot{\alpha}} \quad (2.1)$$

where λ goes over $2N + 2k$ indices and the $2k$ indices are split into $i = 1, \dots, k$ and $\alpha, \dot{\alpha} = 1, 2$ and $x_{\alpha \dot{\alpha}} = x_\mu \sigma_{\alpha \dot{\alpha}}^\mu$. The matrix Δ satisfies the constraint

$$(\bar{\Delta} \Delta)_{ij \dot{\beta}}^{\dot{\alpha}} = \delta_{\dot{\beta}}^{\dot{\alpha}} f_{ij}^{-1} \quad (2.2)$$

¹We use the same notation as ref. [2]

where f_{ij} is a symmetric matrix. Another ingredient in the construction is U which is a $(2N + 2k) \times 2N$ matrix and satisfies the orthogonality relation $\bar{\Delta}U = \bar{U}\Delta = 0$ and $\bar{U}U = 1$. The projection operator is $U\bar{U} = 1 - \Delta f \bar{\Delta}$. The instanton gauge potential $v_\mu = \bar{U}\partial_\mu U$ gives a self dual field strength of instanton number k .

We can now go to the $Sp(N)$ case by projecting by Ω which implies that Δ and U satisfy:

$$\Omega_{N+k}\Delta = \Delta^*\Omega_k, \quad \Omega_{N+k}U = U^*\Omega_N \quad (2.3)$$

where $*$ denotes complex conjugation and Ω_r is a $2r \times 2r$ matrix with r diagonal blocks of σ_2 each.

It is clear that Δ is defined up to the action of constant $Sp(N+k)$ from the left and $GL(k, R)$ from the right. $Sp(N+k)$ acts simultaneously on U on the left. U on the other hand admits local $Sp(N)$ transformations on the right. Using the freedom on Δ we can bring the matrix b in 2.1 to the form

$$b = \begin{pmatrix} 0_{2N \times 2k} \\ 1_{2k \times 2k} \end{pmatrix} \quad (2.4)$$

The residual symmetry of Δ is $Sp(N)$ subgroup of $Sp(N+k)$ left action and $O(k)$ action defined by

$$\Delta \rightarrow \begin{pmatrix} 1_{2N \times 2N} & 0_{2N \times 2k} \\ 0_{2k \times 2N} & g \times 1_{2 \times 2} \end{pmatrix} \Delta g^t \times 1_{2 \times 2} \quad (2.5)$$

with g being an $O(k)$ element. This residual $Sp(N) \times O(k)$ symmetry will play an important role in the next section when we describe the ADHM data using (-1) -, 3 -, 7 -brane system.

Writing $a = \begin{pmatrix} w_{2N \times 2k} \\ a'_{2k \times 2k} \end{pmatrix}$, the constraint on $\bar{\Delta}\Delta$ then implies that a' is a symmetric matrix (i.e. $a'_{(\alpha i)(\dot{\alpha} j)} = a'_{(\alpha j)(\dot{\alpha} i)}$) and therefore transforms as the second rank symmetric tensor under $O(k)$. The constraint also implies that

$$D_{ij}^c \equiv tr_2 \tau^c (\bar{w}w + \bar{a}'a')_{ij} = 0 \quad c = 1, 2, 3. \quad (2.6)$$

where τ^c are Pauli matrices. Note that D_c is a $k \times k$ antisymmetric matrix and transforms in the adjoint representation of $O(k)$. D_c will play the role of the D-terms in the $O(k)$ gauge theory of (-1) -brane instantons. It is also clear that $w^{\dot{\alpha}}$ transforms as a bi-fundamental of $Sp(N) \times O(k)$ and the superscript on w indicates that it transforms as a doublet of $SU(2)_{\dot{\alpha}}$. These fields will play the role of (-1) -brane - 3 -brane states.

2.2 Adjoint and antisymmetric fermion zero modes

Now let us turn to the fermions. Their zero modes can also be obtained by projecting the zero modes for $\mathcal{N} = 4$ $U(2N)$ theory. The result for the $Sp(N)$ adjoint fermions is

$$(\lambda_\alpha^A)_{uv} = \bar{U}_u^\lambda \mathcal{M}_{\lambda i}^A f_{ij} \bar{b}_{\alpha j}^\rho U_{\rho v} - \bar{U}_u^\lambda b_{\lambda i \alpha} f_{ij} (\mathcal{M}_j^T)^{\rho A} U_{\rho v} \quad (2.7)$$

where \mathcal{M}^A is a constant $(2N + 2k) \times k$ matrix of Grassmann variables. Writing

$$\mathcal{M}_i^A = \begin{pmatrix} \mu_{ii}^A \\ \mathcal{M}'_{\beta\ell i}^A \end{pmatrix} \quad (2.8)$$

the constraints on \mathcal{M} are:

$$\begin{aligned} (F_{\dot{\alpha}}^A)_{ij} &\equiv (\mathcal{M}^T)_i^{\lambda A} a_{\lambda\dot{\alpha}j} + \bar{a}_{i\dot{\alpha}}^{\lambda} \mathcal{M}_{\lambda j}^A = 0 \\ (\mathcal{M}'^T)_{\alpha}^A &= \mathcal{M}_{\alpha}^A \end{aligned} \quad (2.9)$$

In particular this implies that \mathcal{M}' transforms in the symmetric representation of $O(k)$. Similarly the zero mode expressions for the fermions λ^Y in the anti-symmetric representation of $Sp(N)$ are given by the above expressions with the collective coordinates \mathcal{M}^A replaced by \mathcal{M}^Y and $(\mathcal{M}^T)^A$ replaced by $-(\mathcal{M}^T)^Y$. This in particular implies that \mathcal{M}'^Y transforms in the anti-symmetric representation of $O(k)$.

The fermionic constraints $F_{\dot{\alpha}}^A$ and $F_{\dot{\alpha}}^Y$ transform in the adjoint and the symmetric representations of $O(k)$ respectively. Note that \mathcal{M}'_{α}^A and $F_{\dot{\alpha}}^A$ carry $(+1)$ charge under $SO(2)$ while \mathcal{M}'_{α}^Y and $F_{\dot{\alpha}}^Y$ carry (-1) charge. As we shall see in the next section, this is exactly the structure one finds in the (-1) -, 3-, 7- brane system.

Finally we will need the explicit expressions of the four supersymmetric and four super-conformal exact zero modes. They are given respectively by choosing

$$\mathcal{M}_i^A = \begin{pmatrix} 0 \\ \delta_{\ell i} \psi_{\beta}^A \end{pmatrix} \quad (2.10)$$

and

$$\mathcal{M}_i^A = \begin{pmatrix} w_{ui} \dot{\zeta}_{\dot{\alpha}}^A \\ 0 \end{pmatrix} \quad (2.11)$$

2.3 Adjoint and anti-symmetric scalars

In the $U(2N)$ theory the adjoint scalar φ^{AB} transforms as a vector under the $SO(6)_R$ symmetry. It satisfies the equation of motion

$$D^2 \varphi^{AB} = \sqrt{2}i[\lambda^A, \lambda^B]. \quad (2.12)$$

The solution is given by:

$$i\varphi^{AB} = -\frac{1}{2\sqrt{2}}\bar{U}(\mathcal{M}^A f \bar{\mathcal{M}}^B - \mathcal{M}^B f \bar{\mathcal{M}}^A)U + \bar{U} \begin{pmatrix} 0_{2N \times 2N} & 0_{2N \times 2k} \\ 0_{2k \times 2N} & \Phi_{k \times k}^{AB} \times 1_{2 \times 2} \end{pmatrix} U \quad (2.13)$$

where Φ^{AB} is the collective coordinate that satisfies the equation

$$L \cdot \Phi^{AB} = \Lambda^{AB} \quad (2.14)$$

with Λ^{AB} and the operator L defined as:

$$\begin{aligned} L \cdot \Phi^{AB} &= \frac{1}{2} \{ \Phi^{AB}, W^0 \} + [a'_\mu, [a'_\mu, \Phi^{AB}]] \\ \Lambda^{AB} &= \frac{1}{2\sqrt{2}} (\bar{\mathcal{M}}^A \mathcal{M}^B - \bar{\mathcal{M}}^B \mathcal{M}^A) \end{aligned} \quad (2.15)$$

where $W^0 = \text{tr}_2 \bar{w}w$. Splitting again $SU(4)_R$ symmetry into $SU(2)_A \times SU(2)_Y \times SO(2)$ and doing the $\Omega \cdot Z_2$ projection, which projects φ^{AY} and φ^\pm into respectively anti-symmetric and adjoint representation of $Sp(N)$ we find that the collective coordinates Φ^{AY} and Φ^\pm transform as symmetric and adjoint representations of $O(k)$ respectively. Also as a result of the projection, $\bar{\mathcal{M}}$ in the above expressions are replaced by $(\mathcal{M}^T)^A$ and $-(\mathcal{M}^T)^Y$.

2.4 Fundamental fields

As we mentioned, the theory we are considering also includes hypermultiplets that transform as bi-fundamental of $Sp(N) \times O(8)$. In fact we shall be precisely interested in computing the correlation function of the $O(8)$ flavour currents. The fundamental fermion zero mode is

$$\eta_u^{\alpha r} = \bar{U}_u^\lambda b_{\lambda i}^\alpha f_{ij} \mathcal{K}_j^r \quad (2.16)$$

where r is the $O(8)$ flavour index and \mathcal{K} 's are Grassmann numbers that transform as bi-fundamental of $O(k) \times O(8)$. Note that \mathcal{K} does not transform under $SO(4)_I \times SO(4)_E$. Since the spinor $\eta_u^{\alpha r}$ couples to the scalar φ^+ through the term $\text{tr}(\varphi^+ \eta^{\alpha r} \eta_\alpha^r)$, then the equation of motion for φ^+ has an additional term $\eta^{\alpha r} \eta_\alpha^r$ on the right hand side of (2.12). The effect of this term is the addition of a term

$$(\Lambda_f)_{ij} = \mathcal{K}_i^r \mathcal{K}_j^r \quad (2.17)$$

to the right hand side of equation (2.14) but only for the component Φ^+ . In summary, after the projection of the $\mathcal{N} = 4$ $U(2N)$ result, which contains those fields of $\mathcal{N} = 2$ $Sp(N)$ which are in the adjoint and antisymmetric reps, the inclusion of the hypers in the fundamental representation is just through the above modification of the constraint for Φ^+ .

2.5 Multi-instanton measure

Now we can write down the measure for k -instantons. It is just integration over the collective coordinates discussed above together with the constraints (eqns. (2.6), (2.9) and (2.14)):

$$\begin{aligned} \int da' dw d\mu^A d\mu^Y d\mathcal{M}'^A d\mathcal{M}'^Y d\Phi^{AY} d\Phi^+ d\Phi^- d\mathcal{K} \\ \delta(F_\alpha^A) \delta(F_\alpha^Y) \delta(D^c) \delta(L \cdot \Phi^{AB} - \Lambda_{\text{tot}}^{AB}) \exp\left(-\frac{1}{g^2} \Phi \cdot \Lambda_{\text{tot}}\right) \end{aligned} \quad (2.18)$$

where $\Lambda_{\text{tot}} = \Lambda + \Lambda_f$. For later use, it is convenient to replace the integration over Φ^{AY} and Φ^a by

$$\begin{aligned} \int d\Phi^{AY} d\Phi^+ d\Phi^- \delta(L \cdot \Phi^{AB} - \Lambda_{\text{tot}}^{AB}) \exp(\Phi \cdot \Lambda_{\text{tot}}) = \\ \int db^{AY} db^+ db^- \exp(-B^{AB} \cdot L \cdot B_{AB} + \frac{i}{g} B \cdot \Lambda_{\text{tot}}) \end{aligned} \quad (2.19)$$

Where B^{AB} is a $4k \times 4k$ matrix whose elements b_{ij}^{AY} and b_{ij}^a are respectively symmetric and adjoint reps of $O(k)$.

Before analyzing the large N limit of this measure, it will be instructive to make a comparison of the above measure with the one for $D(-1)$ -branes in the presence of 3-branes and 7-branes. This we will do in the next section.

3 ADHM measure and D-instantons

In the last section, we discussed the ADHM measure for the collective coordinates of the k instantons in the $Sp(N)$ gauge theory with the specific matter content appearing on the type I' 3-branes. In this section we will consider the Higgs branch of a system of k (-1) -branes, N 3-branes and 8 of the 7-branes on an orientifold plane and will see that, as it is expected [21, 22], it gives the same measure as the ADHM analysis of the last section. The field content appearing on the world volume of the (-1) -branes is

D(-1)-brane World-volume content

Bosons	Fermions	$SO(4)_E$	$SO(4)_I$	$SO(2)$	$SO(k)$	$Sp(N)$	$O(8)$
b^{AY}		(2, 2)	(1, 1)	0	$\frac{1}{2}k(k+1)$	1	1
b^a		(1, 1)	(1, 1)	± 2	$\frac{1}{2}k(k-1)$	1	1
$a'_{\alpha\dot{\alpha}}$		(1, 1)	(2, 2)	0	$\frac{1}{2}k(k+1)$	1	1
	$\Psi^{\dot{\alpha}Y}$	(1, 2)	(1, 2)	+1	$\frac{1}{2}k(k+1)$	1	1
	$\Lambda^{\dot{\alpha}A}$	(2, 1)	(1, 2)	-1	$\frac{1}{2}k(k-1)$	1	1
	\mathcal{M}'^A_{α}	(2, 1)	(2, 1)	+1	$\frac{1}{2}k(k+1)$	1	1
	\mathcal{M}'^Y_{α}	(1, 2)	(2, 1)	-1	$\frac{1}{2}k(k-1)$	1	1
$w^{\dot{\alpha}}$		(1, 1)	(1, 2)	0	k	$2N$	1
	μ^Y	(1, 2)	(1, 1)	-1	k	$2N$	1
	μ^A	(2, 1)	(1, 1)	+1	k	$2N$	1
	\mathcal{K}	(1, 1)	(1, 1)	-1	k	1	8

Here the fields are classified according to their representations under $SO(k)$ gauge theory of the world volume action of the (-1) -branes, in which the $Sp(N)$ and $O(8)$ gauge theories of the 3-branes and 7-branes respectively are the global symmetries. In addition they carry specific representations of the $SO(4)_E \times SO(4)_I \times SO(2)$ which are the rotation groups with E denoting the 4 directions transversal to 3-branes and longitudinal to 7-branes, I denoting the 4 directions inside 3-branes world volume and $SO(2)$ denotes

the two transversal direction to the 7-branes. As before, the indices A and Y refer to the two $SU(2)$'s in $SO(4)_E$ and α and $\dot{\alpha}$ refer to the two $SU(2)$'s of $SO(4)_I$. It will be useful to make comparison with the field content for the (-1) -, 3- brane system in type IIB that is relevant for the $\mathcal{N} = 4$ theory. First of all, we have additional fermionic fields \mathcal{K} transforming as bi-fundamentals of $SO(k) \times O(8)$ arising from open strings between (-1) and 7-branes. These describe the collective coordinates of the hyper multiplets in the $(2N, 8)$ of $Sp(N) \times O(8)$. Secondly, the adjoint fields of the $U(k)$ gauge theory in IIB splits into adjoint and symmetric representations of $SO(k)$ under the $\Omega \cdot Z_2$ -projection as indicated above. Finally the (-1) -3-brane states satisfy a reality condition;

$$\bar{w}_u^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{uv} w_v^{\dot{\beta}} \quad (3.1)$$

The (-1) -brane action for these fields is

$$S = \left(\frac{1}{g_{-1}^2} S_G + S_K + S_D \right) \quad (3.2)$$

where

$$\begin{aligned} S_G &= \text{tr}_k \{ [b^{AY}, b^{BX}]^2 + [b^a, b^{a'}]^2 + [b^{AY}, b^a]^2 + \Lambda^{\dot{\alpha}A} [b_{AY}, \Psi_{\dot{\alpha}}^Y] + 2|D|^2 \} \\ S_K &= \text{tr}_k \{ [b^{AY}, a'_{\alpha\dot{\alpha}}]^2 + [b^a, a'_{\alpha\dot{\alpha}}]^2 + b^{AY} w_u^{\dot{\alpha}} w_{\dot{\alpha}}^u b_{AY} + b^a w_u^{\dot{\alpha}} w_{\dot{\alpha}}^u b^a + \mathcal{M}'^Y_{\alpha} [b_{AY}, \mathcal{M}'^{\alpha A}] \\ &\quad + \mathcal{M}'^Y_{\alpha} [b^-, \mathcal{M}'^{\alpha}_Y] + \mathcal{M}'^A_{\alpha} [b^+, \mathcal{M}'^{\alpha}_A] \} + \mu^A b_{AY} \mu^Y + \mu^A b^- \mu_A + \mu^Y b^+ \mu_Y + \mathcal{K} b^+ \mathcal{K} \\ S_D &= \text{tr}_k \{ [a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha A}] \Lambda^{\dot{\alpha}A} + [a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha Y}] \Psi_{\dot{\alpha}}^Y + \mu^{Yu} w_u^{\dot{\alpha}} \Psi_{\dot{\alpha}Y} + \mu^{Au} w_u^{\dot{\alpha}} \Lambda_{\dot{\alpha}A} \\ &\quad + D_{\dot{\beta}}^{\dot{\alpha}} (w_{\dot{\alpha}}^u w_u^{\dot{\beta}} + \bar{a}'^{\dot{\beta}\alpha} a'_{\alpha\dot{\alpha}}) \} \end{aligned} \quad (3.3)$$

Here $D_{\dot{\alpha}\dot{\beta}}$ is an $SU(2)$ triplet of auxiliary fields in the adjoint of $SO(k)$. In the infrared limit $\frac{1}{g_{-1}^2} (= \alpha'^2) \rightarrow 0$, S_G becomes irrelevant and therefore we shall drop this term. Note that the remaining action $S_D + S_K$ has the following scaling invariance:

$$\begin{aligned} (w, a') &\rightarrow \kappa(w, a') \\ (b^{AY}, b^a) &\rightarrow \kappa^{-1}(b^{AY}, b^a) \\ (\mu^A, \mu^Y) &\rightarrow \kappa^{1/2}(\mu^A, \mu^Y) \\ (\Lambda, \Psi) &\rightarrow \kappa^{-3/2}(\Lambda, \Psi) \\ (\mathcal{M}'^A, \mathcal{M}'^Y) &\rightarrow \kappa^{1/2}(\mathcal{M}'^A, \mathcal{M}'^Y) \\ \mathcal{K} &\rightarrow \kappa^{1/2} \mathcal{K} \end{aligned} \quad (3.4)$$

Moreover one can easily see that the integration measure is also invariant under the above scaling. From S_D we obtain the D -term constraints. Explicitly, integrating D , Ψ and Λ we find the bosonic constraint:

$$(W^c + (\sigma^c)_{\dot{\beta}}^{\dot{\alpha}} (\bar{a}' a')_{\dot{\alpha}}^{\dot{\beta}})_{[ij]} = 0 \quad (3.5)$$

and the fermionic constraints:

$$\begin{aligned}(\mu_u^Y w_\alpha^u + [\mathcal{M}'^{\alpha Y}, a'_{\alpha\dot{\alpha}}])_{(ij)} &= 0 \\(\mu_u^A w_\alpha^u + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot{\alpha}}])_{[ij]} &= 0\end{aligned}\tag{3.6}$$

where W^c is defined through the equation

$$W_{ij}^0 \delta_{\dot{\beta}}^{\dot{\alpha}} + (\sigma^c)_{\dot{\beta}}^{\dot{\alpha}} W_{ij}^c = \epsilon^{uv} w_{ui}^{\dot{\alpha}} w_{\dot{\beta}vj}\tag{3.7}$$

Note that W^0 and W^c are respectively symmetric and antisymmetric in ij .

It is now clear how the correspondence between the fields on the k (-1) -branes and the collective coordinates of the k instantons of previous section goes. The bosonic fields $a'_{\alpha\dot{\alpha}}$ and $w_{\dot{\alpha}}^{\dot{\alpha}}$ correspond to the two components of the bosonic collective coordinate $a_{\lambda i \dot{\alpha}}$. The other bosonic fields b^{AY} and b^a correspond to the coordinates introduced with the same notation in (2.19). The fermions \mathcal{M}'^A , \mathcal{M}'^Y , μ^A , μ^Y and \mathcal{K} correspond to the Grassmann coordinates introduced with the same notation in section 2 when we discussed the zero modes of adjoint, antisymmetric and fundamental fermions. Moreover we see that the constraints in (3.5) and (3.6) are exactly the same as the ones given in last section and the action S_K for the fields just mentioned is the the same as the action in (2.19).

In order to integrate the delta function constraints (3.5) and (3.6), we decompose the fermions χ 's in terms of components ξ 's and ν 's parallel and orthogonal to w respectively:

$$\begin{aligned}\mu_{ui}^Y &= (\xi_{\dot{\alpha}}^Y)_{ij} w_{uj}^{\dot{\alpha}} + \nu_{ui}^Y \\ \mu_{ui}^A &= (\xi_{\dot{\alpha}}^A)_{ij} w_{uj}^{\dot{\alpha}} + \nu_{ui}^A\end{aligned}\tag{3.8}$$

The integration over the $2 \times 4k(N - k)$ fermionic variables (ν^Y, ν^A) can be easily done since they appear in the action as $\frac{1}{g}(\nu^A b^{AY} \nu^Y + \nu^A b^- \nu^A + \nu^Y b^+ \nu^Y)$. This then gives

$$g^{-4k(N-k)} (\det_{4k \times 4k}(B))^{N-k}\tag{3.9}$$

The integration over $4k^2$ out of $2 \times (4k^2)$ fermionic variables (ξ^Y, ξ^A) can be done using the delta-function constraints (3.6), which simply gives $(\det_{2k \times 2k} W)^{2k}$. Note that this factor cancels exactly $(\det_{2k \times 2k} W)^{-2k}$ coming from the Jacobian of the transformations (3.8). The remaining ξ integrations are over $2k(k - 1)$ variables, ξ^Y 's, and $2k(k + 1)$ variables, ξ^A 's.

For the integration over $4kN$ bosonic variables $w_{ui}^{\dot{\alpha}}$, as in ref.[2], since the integrand is only dependent on the gauge invariant matrix W , it is convenient to perform the angular integrations. This can be done following the method explained in [2]. The steps involve firstly to bring w into a standard upper triangular form by using the gauge invariance.

In our case, using $Sp(N)$ gauge transformation:

$$w = \begin{pmatrix} v_1^1 & v_1^2 & . & . & v_1^k \\ 0 & v_2^2 & . & . & v_2^k \\ . & . & . & . & . \\ 0 & 0 & . & . & v_k^k \\ 0 & 0 & . & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & 0 \end{pmatrix} \quad (3.10)$$

where the right hand side is an $N \times k$ matrix with quaternionic entries and the diagonal entries v_i^i are the identity elements of the quaternion. This change of variables and the subsequent angular integration gives the following result:

$$d^{4kN} w \rightarrow Vol\left(\frac{Sp(N)}{Sp(N-k)}\right) \prod_{j=1}^k (v_j^j)^{4N-4j+3} d^{2k(k-1)} v \quad (3.11)$$

where the volume factor appearing above is the product $\prod_{j=1}^k Vol(S^{4N-4j+3})$. Changing further the variables v to gauge invariant variables W defined in (3.7) we find that $\int d^{4kN} w$ is replaced by

$$Vol\left(\frac{Sp(N)}{Sp(N-k)}\right) \int d^{k(k+1)/2} W^0 \prod_{c=1,2,3} d^{k(k-1)/2} W^c (\det_{2k \times 2k} W)^{N-k+1/2}$$

Moreover the integration over W^c can be done simply by using the bosonic D-term Delta function which then amounts to replacing W^c by $(\sigma^c)_{\dot{\alpha}}^{\dot{\beta}} (\bar{a}' a')^{\dot{\alpha}}_{\dot{\beta}}$.

Finally taking into account the factor g^{4Nk+4k} coming from the normalization of zero mode wave functions [23], we are left with the following integration measure

$$c(g, k, N) \int d^{k(k+1)/2} W^0 d^{2k(k+1)} b^{AY} d^{k(k-1)} b^a d^{2k(k+1)} a' d^{2k(k+1)} \mathcal{M}'^A d^{2k(k-1)} \mathcal{M}'^Y d^{8k} \mathcal{K} d^{2k(k+1)} \xi^A d^{2k(k-1)} \xi^Y (\det_{2k \times 2k} W^0)^{N-k+1/2} (\det_{4k \times 4k} B)^{N-k} \exp(-S') \quad (3.12)$$

where

$$S' = \text{tr}_k \{ [B^{AB}, a'_{\alpha\dot{\alpha}}]^2 + B^{AB} W^0 B^{AB} + \frac{i}{g} B^{AB} \Upsilon^{AB} \} + \frac{i}{g} \mathcal{K} b^+ \mathcal{K} \\ c(g, k, N) = g^{4k(k+1)} Vol\left(\frac{Sp(N)}{Sp(N-k)}\right) \quad (3.13)$$

Here

$$\Upsilon^{AB} = \zeta^A W \zeta^B + \mathcal{M}'^A \mathcal{M}'^B \quad (3.14)$$

where

$$\zeta^{\mathcal{A}} = \begin{pmatrix} \xi^Y \\ \xi^A \end{pmatrix}$$

and

$$\mathcal{M}'^{\mathcal{A}} = \begin{pmatrix} \mathcal{M}'^Y \\ \mathcal{M}'^A \end{pmatrix}$$

The above action S' , apart from the term $\mathcal{K}b^+\mathcal{K}$, has exactly the same form as the one appearing in equation (4.55) of ref. [2]. The difference is that in the present case, due to the Ω -projection, the components of the variables B^{AB} , $\zeta^{\mathcal{A}}$ and $\mathcal{M}'^{\mathcal{A}}$ belong to different representations of $SO(k)$. Thus the large N , saddle point analysis of [2] can be adapted to our case, after keeping track of the above projection. Therefore it will suffice to state the steps involved and then give the final solution. One first makes the rescaling $B \rightarrow \sqrt{N}B$ and raises the two determinants in (3.12) to an effective action keeping only leading terms of order N . A class of solution to the saddle point equations, modulo the $SO(k)$ gauge transformation, is obtained by setting to zero all the $SO(k)$ adjoint variables b^\pm , with the symmetric ones, W^0 , b^{AY} and $a'_{\alpha\dot{\alpha}}$ being diagonal. This corresponds to the k instantons having independent positions in X_μ^i (in R^4) and sizes ρ^i with $i = 1, \dots, k$ and $\mu = 1, \dots, 4$, given by the diagonal elements of $a'_{\alpha\dot{\alpha}}$ and W^0 respectively. Moreover, the diagonal elements of b^{AY} give the positions Ω_a^i ($a = 1, 2, 3$) of the instantons on S^3 . In other words, each instanton is parametrized by a point $(X_n^i, \rho^i, \Omega_a^i)$ of $AdS_5 \times S^3$.

Actually in the present case this is not the whole story. We have also another class of saddle point solutions which correspond to pairs of instantons splitting off the 7-brane world volume and sitting at the mirror points in the transverse directions to the 7-brane. To illustrate this let us consider $k = 2r$ be an even number. in this case we can take b^\pm to be diagonal $(r \times r)$ matrix times σ_2 and b^{AY} , $a'_{\alpha\dot{\alpha}}$ and W^0 to be diagonal $(r \times r)$ matrix times $1_{2 \times 2}$ subject to the condition

$$b^{(+}b^{-)} + b^{AY}b_{AY} = (W^0)^2 \quad (3.15)$$

If the r eigenvalues are different then the $O(2r)$ gauge group is broken to the semidirect product $S_r \ltimes (Z_2)^r$ where S_r is the permutation of the r eigenvalues and $(Z_2)^r$ acts by reflecting the eigenvalues of b^\pm (i.e. $b^\pm \rightarrow -b^\pm$). Equation (3.15), together with the gauge group $S_r \ltimes (Z_2)^r$ then shows that this saddle point configuration corresponds to r instantons on $AdS_5 \times S^5/Z_2$ (In the covering space $AdS_5 \times S^5$ there are $2r$ instantons at the image points).

Clearly there are also other saddle point solutions intermediate to the above two cases where some of the instantons stay in the 7-brane world volume and others split off the 7-brane in pairs. Moreover some of the eigenvalues may also coincide. But in all cases

the saddle point solution has a clear interpretation of k instantons forming subsystems of single or "bound" instantons, some of them living in the 7-brane world volume and others living in the bulk $AdS_5 \times S^5/Z_2$ ²

4 Exact saddle points in the large-N limit

It has been argued in [2], in $\mathcal{N} = 4$ context, that, after taking into account the fluctuations around the saddle point, the leading contribution to the integral, in the large N limit, comes from the most degenerate configuration, corresponding to the k instantons being of the same size and in the same position in R^4 and S^5 , i.e. from the k -instanton bound state in $AdS_5 \times S^5$. The argument rests on the analysis of the measure when two of the eigenvalues become equal. In this limit some of the previously massive variables become massless. The quadratic determinant of the bosonic variables therefore diverges in this limit. However in a supersymmetric theory the integration over the would-be massless fermionic variables should cancel this divergence. To see this let us consider $k = 2$ case in the $\mathcal{N} = 2$ model under consideration (of course the same consideration will also apply to the $\mathcal{N} = 4$ model considered in [2]). There are two possible generic saddle point solutions:

$$\begin{aligned} 1) \quad & b^\pm = 0, \quad \text{and } b^{AY}, a'_{\alpha\dot{\alpha}}, W^0 \text{ diagonal with different eigenvalues,} \\ 2) \quad & b^\pm = x\sigma_2, \quad \text{and } b^{AY}, a'_{\alpha\dot{\alpha}}, W^0 \text{ proportional to } 1_{2 \times 2} \end{aligned} \quad (4.1)$$

In the first case the $O(k) = O(2)$ is broken to Z_2 while in the second case it is unbroken. In the first case, to be explicit let us assume that the real part of only b^{AY} for $A = Y = 1$ has different eigenvalues and all the other variables have equal eigenvalues. Then the off-diagonal components of $b^{AY}; (A, Y) \neq (1, 1)$, $a'_{\alpha\dot{\alpha}}$, b^\pm , \mathcal{M} 's and ξ 's and the imaginary part of b^{11} , acquire masses that vanish in the limit the two eigenvalues coincide (note that a certain combination of the off-diagonal components of W^0 and other fields have quadratic terms as follows from equation 6.6 that does not vanish in the limit of coinciding eigenvalues). Let M be the difference of the eigenvalues of the real part of b^{11} . We would now try to extract the dependence of the measure on M . At first sight it may appear that there are 9 off-diagonal bosons and only 16 real off-diagonal fermions whose masses vanish in this limit, and therefore the integral over the massive variables diverges as $M^{-9+\frac{16}{2}} = M^{-1}$. However by using $O(2)$ transformation one can always bring the real part of b^{11} into diagonal form, with the Jacobian of the transformation being $d^3b^{11} \rightarrow d^2b^{11}M$ modulo the volume of $O(2)$, where on the left hand side the 3 b^{11} 's are in the symmetric representation of $O(2)$ while on the right

²In the case of an $\mathcal{N} = 4$ $Sp(N)$ gauge theory the coordinates (b^{AY}, b^\pm) are replaced by a six-component vector of variables in the antisymmetric of $O(k)$. The analog of (3.15) would then give, together with the other bosonic coordinates, the moduli space $(AdS_5 \times RP^5)^r$ (for $k = 2r$), in agreement with [24].

hand side we have the 2 diagonal b^{11} 's. In the total measure therefore M cancels as is expected from supersymmetry.

In the second case, the fact that the measure is independent of the mass is even more straightforward to see. b^\pm , \mathcal{M}'^Y and ξ^Y in this case remain massless and the integration over the massive bosonic b^{AY} and $a'_{\alpha\dot{\alpha}}$ and fermionic \mathcal{M}'^A , ξ^A and \mathcal{K} cancel exactly as expected from supersymmetry.

The argument above can be easily extended to other values of k and for all the possible saddle point solutions corresponding to subsystems of instantons in the 7-brane world volume and in the full 10-dimensions. It is also easy to see that the same argument applies to the $\mathcal{N} = 4$ case considered in [2]; there is no divergence in the measure when the two eigenvalues coincide.

What then selects the saddle point in [2] corresponding to collapse of all the instantons to a common position and scale? The answer is that in all the other saddle points there are more fermion zero modes than the 16 corresponding to the supersymmetric and superconformal zero modes. Indeed if there are subsystems of instanton 'bound states', there would be center of mass of each of these subsystems resulting in more fermion zero modes. In fact these extra fermion zero modes are the ones contained in \mathcal{M}' and ζ (part of μ which is parallel to w) in the decomposition of the fermion collective coordinates $\mathcal{M} = \begin{pmatrix} \mu \\ \mathcal{M}' \end{pmatrix}$. In other words these extra zero modes are not contained in ν and $\bar{\nu}$ that are parts of μ and $\bar{\mu}$ that are orthogonal to \bar{w} and w respectively. If one considers Yang-Mills correlation functions involving operators that contain only 16 fermions then clearly these other saddle points cannot contribute. On the other hand if one computes correlators involving more than 16 fermions, then as shown in [2], the leading order term in large N limit comes when all the extra fermions (other than 16) appear as ν and $\bar{\nu}$ due to the fact that they transform as fundamental of $SU(N)$ and the $SU(N)$ trace gives a factor of N for each pair of ν and $\bar{\nu}$. Thus to the leading order in N only the saddle point corresponding to all the instantons at same position and of same scale contribute.

In the $\mathcal{N} = 2$ case under consideration the same argument applies. For any Yang-Mills correlators, the leading order in N arises when one takes the maximum possible fermions in the ν and $\bar{\nu}$ part of the fermion collective coordinates. Of course at any saddle point there are at least 8 exact fermion zero modes coming from \mathcal{M}' and ζ . Precisely 8 exact zero modes appear when all the instantons sit in the 7-brane world volume at a common position and with common size. Thus Yang-Mills correlators that contain only 8 fermion collective coordinates \mathcal{M} (i.e. other than the $O(8)$ fermion collective coordinates \mathcal{K}) will get contribution only from the saddle point corresponding to the 'bound state' of all the instantons on the 7-brane world volume. In the next section we will consider such a correlation function involving four $O(8)$ currents. However one may also consider correlators involving more than 8 fermions. If the extra fermions

can appear as ν and $\bar{\nu}$ then the leading order in N is again given by the above saddle point. But as we shall see in section 5, for correlators involving operators that couple to Kaluza-Klein modes of closed string states on S^5/Z_2 this leading order in N vanishes, and one gets contribution from saddle points (for even $k = 2r$) that correspond to r instantons sitting in $AdS_5 \times S^5/Z_2$. Anticipating these results, in the remaining part of this section we analyse these two saddle points in detail, the one where all the instantons sit at a common point in the 7-brane and second (for even $k = 2r$) when r instantons sit in the bulk $AdS_5 \times S^5/Z_2$.

In the first case $b^\pm = 0$ and all the b^{AY} , $a'_{\alpha\dot{\alpha}}$ and W^0 are proportional to identity matrix with the constraint $b^{AY}b_{AY} = (W^0)^2$. One can then perform the analysis of small fluctuations around this configuration and arrive at the following measure for the collective coordinates of k instantons:

$$Ng^4 \int \rho^{-5} d\rho d^4 X d^3 \Omega \prod_{A=1,2} d^2 \xi^A d^2 \psi^A \mathcal{Z}_k \quad (4.2)$$

where \mathcal{Z}_k is the partition function of the type-I' k D-instantons in the presence of 8 7-branes at an orientifold 7-plane:

$$\begin{aligned} \mathcal{Z}_k = \int dX db^a d\tilde{\Theta} d\Theta d\mathcal{K} \exp(-\text{tr}_k([X_i, X_j]^2 + [b^a, X_i]^2 + [b^a, b^{a'}]^2 \\ + \tilde{\Theta}[b^-, \tilde{\Theta}] + \Theta[b^+, \Theta] + \Theta\Gamma^i[X_i, \tilde{\Theta}]) + \mathcal{K}b^+\mathcal{K}), \end{aligned} \quad (4.3)$$

where we have assembled b^{AY} and $a'_{\alpha\dot{\alpha}}$ into an 8-vector X_i , $i = 1, \dots, 8$, of variables in the traceless, symmetric representation of $SO(k)$, after a rescaling by $N^{\frac{1}{4}}$ for each bosonic variable. Θ and $\tilde{\Theta}$ are chiral $SO(8)$ spinors of opposite chirality in the traceless, symmetric and adjoint representation of $SO(k)$ respectively. They are obtained respectively by combining ξ^A with \mathcal{M}'^A and ξ^Y with \mathcal{M}'^Y also taking into account a rescaling factor $N^{\frac{1}{8}}g^{\frac{1}{2}}$ for each fermionic variable. In obtaining the factor N in 4.2 we have taken into account a factor $N^{-2kN+k(k-1/2)}$ from $Vol(\frac{Sp(N)}{Sp(N-k)})$, and $N^{-k(k+1)/4}$ from W^0 integration. We should stress that in obtaining the action of 4.3, first one needs to choose a specific basis for the $SO(8)$ gamma-matrices, given by

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \times \sigma_1 \times \sigma_1 & \mu = 1, 2, 3, 4 \\ \Gamma_5 &= \gamma_5 \times \sigma_1 \times \sigma_1 & \Gamma_6 = 1 \times \sigma_2 \times \sigma_1 \\ \Gamma_7 &= 1 \times \sigma_3 \times \sigma_1 & \Gamma_8 = 1 \times 1 \times \sigma_2 \end{aligned} \quad (4.4)$$

Here $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$ and $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the four dimensional (4×4) matrices corresponding to $SO(4)_I$. The first set of sigma matrices corresponds to $SU(2)_R$. In this basis the 8 dimensional chirality operator $\Gamma_9 = 1 \times 1 \times \sigma_3$ and the charge conjugation operator is $C = c \times \sigma_2 \times \sigma_3$ with $c = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ being the 4-dimensional charge conjugation operator. Secondly, in order to bring the interaction term involving X_i , Θ

and $\tilde{\Theta}$ to the form appearing in (4.3), one has to perform an Ω -dependent rotation on the above gamma-matrices.

In the second case (for $k = 2r$), $b^\pm = M1_{r \times r} \times \sigma_2$ and b^{AY} , $a'_{\alpha\dot{\alpha}}$ and W^0 proportional to identity with the constraint eq.(3.15). After gauge fixing and integrating over the massive fields, the dependence on M disappears as discussed above, and the action for the massless variables can again be expanded up to the quartic term. The resulting action has $U(r)$ gauge symmetry and is given by:

$$\sqrt{N}g^8 \int \rho^{-5} d\rho d^4 X d^5 \Omega \prod_{A=1,2;i=\pm} d^2 \xi_i^A d^2 \psi_i^A \mathcal{Z}_r^{II} \quad (4.5)$$

where

$$\begin{aligned} \mathcal{Z}_r^{II} = & \int dX db^a d\tilde{\Theta} d\Theta \exp(-\text{tr}_k([X_i, X_j]^2 + [b^a, X_i]^2 + [b^a, b^{a'}]^2 \\ & + \tilde{\Theta}[b^-, \tilde{\Theta}] + \Theta[b^+, \Theta] + \Theta\Gamma^i[X_i, \tilde{\Theta}]), \end{aligned} \quad (4.6)$$

where all the variables are in the adjoint representation of $SU(r)$ and the Γ matrices are the same as in eq.(4.4). Note that in this case there are 16 fermion zero modes in eq(4.5) and the bosonic integral in (4.5) can be recognized as the integral over $AdS_5 \times S_5/Z_2$, with Z_2 being the Weyl group acting on the saddle point solution $b^\pm \rightarrow -b^\pm$ subject to the condition (3.15) defining S^5 . The quantity \mathcal{Z}_r^{II} in eq.(4.6) is exactly the r type IIB D-instanton matrix theory after having removed the center of mass variables which is given by

$$\mathcal{Z}_r^{II} = r \sum_{m|r} \frac{1}{m^2} \quad (4.7)$$

5 Yang-Mills correlators in the instanton background

After having discussed the ADHM measure and the exact saddle points in large N limit, we now compute some Yang-Mills correlators in the instanton background. By the AdS/CFT correspondence, these correlators will be given by certain bulk terms in the effective action of the string theory living in the bulk. Given the fact that the Yang-Mills correlators will appear with the instanton action factor $e^{2\pi i k \tau_{YM}}$ and the relation between τ_{YM} to the complexified string coupling constant τ in type I', the effective action terms that will contribute are precisely the ones that get contribution from k D-instantons. It is known via heterotic-type I-type I' duality in 8-dimensions, that there are terms like $t_8 F^4$ and $t_8 R^4$ (and their supersymmetric partners) that get contribution from world sheet instantons, D-string instantons and D-instantons in these three dual theories respectively. These terms live in the 7-brane world volume in the type I' theory. However we will argue that there are also some bulk terms in the type I' effective action (i.e. living in 10-dimensions) of the form $t_8 t_8 R^4$ and its supersymmetric partners that receive corrections from D-instantons. We will first compute some correlators in Yang-Mills

that in the large N limit receive contribution from the saddle point corresponding to all instantons sitting on the 7-brane world volume and show that they can be reproduced as coming from the $t_8 F^4$ and $t_8 R^4$ terms in the 7-brane bulk theory. Next we will consider some correlator that selects the saddle point which corresponds to instantons sitting in the bulk $AdS_5 \times S^5/Z_2$ and show that they come from the $t_8 t_8 R^4$ term in the string theory.

5.1 $O(8)$ current correlators and AdS_5 propagators

The $O(8)$ gauge potential couples to the flavour current in the 3-brane world volume theory via the term $A_\mu^a J_\mu^a$ with $J_\mu^a = \bar{q} T^a \vec{D}_\mu q - \bar{q} T^a \overleftarrow{D}_\mu q + \bar{\eta}^{\alpha A} T^a \sigma_{\mu\alpha\dot{\alpha}} \eta_A^{\dot{\alpha}}$ and $D_\mu = \partial_\mu + A_\mu$ is the $Sp(N)$ covariant derivative. On the Yang-Mills side we are interested in computing a 4-point function of the currents J_μ^a in the presence of $Sp(N)$ instantons. We will see below that this is sufficient to soak the 8 exact gaugino zero modes and moreover for odd instantons will soak also the 8 fundamental zero modes giving rise to odd parity $O(8)$ invariant. In the large N saddle point approximation, the correlation function is just given by plugging in the classical solutions of the fields. First let us consider the bosonic part of the $O(8)$ current. The scalar field ϕ satisfies the equation:

$$D^2 q = \lambda \eta \quad (5.1)$$

where λ and η are the gaugino and fundamental fermion fields. Plugging in the expressions for the zero modes of λ and χ as given in equations (2.7) and (2.16) in terms of the ADHM data, we can solve the above equation with the result:

$$q_u^r = \bar{U}_u^\rho \mathcal{M}_{\rho i} f_{ij} \mathcal{K}_j^r \quad (5.2)$$

where the index r and u label the fundamentals of $O(8)$ and $Sp(N)$ respectively, ρ runs over $2N + 2k$ values, and i, j run from 1 to k .

Using now the explicit form for \mathcal{M} for the case of the 8 exact zero modes for the gauginos (2.10), (2.11), and the saddle point solution for which

$$\begin{aligned} f_{ij} &= \frac{1}{y^2 + \rho^2} \delta_{ij} & (a'_\mu)_{ij} &= -X_\mu \delta_{ij} \\ (W^0)_{ij} &= \rho^2 \delta_{ij} & W^c &= 0 \end{aligned}$$

where $y = x - X$, we can express the contribution of the scalars to the current J_μ , after some straightforward algebra, as

$$\begin{aligned} J_\mu &= \frac{1}{N^{1/4} g} \rho^2 f^4 \mathcal{K} T^a \mathcal{K} \quad [\psi^{\alpha A} \psi_{\beta A} y_\nu (\sigma^{[\nu} \bar{\sigma}^{\mu]})_\alpha^\beta + \xi_\alpha^A \xi_A^{\dot{\beta}} \rho^2 (\bar{\sigma}^{[\mu} \sigma^{\nu]})_{\dot{\beta}}^\alpha y^\nu \\ &\quad + 2\psi^{\alpha A} \xi_A^{\dot{\beta}} ((y^2 - \rho^2) \sigma^\mu - 2y^\mu y^\nu \sigma^\nu)_{\alpha\dot{\beta}}] \end{aligned} \quad (5.3)$$

The fermion part of the current $\bar{\eta}^{\alpha A} T^a \sigma_{\mu\alpha\dot{\alpha}} \eta_A^{\dot{\alpha}}$ is a bit more tricky. At first sight it appears that this term cannot soak the 8 exact supersymmetric and superconformal

zero modes. Indeed the zero mode of η^α does not contain \mathcal{M}^A collective coordinates and $\bar{\eta}^{\dot{\alpha}}$ has no zero mode of the Dirac operator. However a closer inspection shows that one needs to solve the classical equation for $\bar{\eta}^{\dot{\alpha}}$ in the presence of the 8 exact zero modes.

$$D_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = (\lambda_\alpha^A q_A + \varphi^+ \eta_\alpha) \quad (5.4)$$

Note that this equation does not have a solution for arbitrary collective coordinates \mathcal{M}^A . The reason for this is that the right hand side of the above equation has a non-zero overlap with the zero modes for η^α given in (2.16) i.e. zero modes of the conjugate operator $\bar{D}_{\dot{\alpha}\alpha}$. This is evident from the $N = 2$ analogue of the equation (3.7) of [2] which reads

$$D_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = (\lambda_\alpha^A q_A + \varphi^+ \eta_\alpha) + \chi_\alpha \quad (5.5)$$

where

$$\bar{\eta}^{\dot{\alpha}} = \frac{1}{2}\bar{U}\mathcal{M}^A f\bar{\Delta}^{\dot{\alpha}}\mathcal{M}_A f\mathcal{K} + \frac{1}{2}\bar{U}\hat{\Phi}^+ f\Delta^{\dot{\alpha}} f\mathcal{K} \quad (5.6)$$

and

$$\chi_\alpha = \bar{U}b_\alpha f\Phi^+ \mathcal{K} \quad (5.7)$$

is a zero mode of the operator $\bar{D}^{\dot{\alpha}\alpha}$ and $\hat{\Phi}^+ = \begin{pmatrix} 0 & 0 \\ 0 & \Phi^+ \times 1_{2 \times 2} \end{pmatrix}$ is the collective coordinate of φ^+ appearing in (2.13). It is clear therefore that χ_α is not in the image of the operator $D_{\alpha\dot{\alpha}}$.

A careful analysis of the collective coordinate approach followed here (as well as in the previous papers [2], and others) shows that in the correlator, in the leading semiclassical approximation $\bar{\eta}^{\dot{\alpha}}$ should be replaced by the right hand side of eq.(5.6). However here we will give a simple argument to prove this which is based on the fact that we need to soak the 8 exact zero modes.

The right hand side of the expression in terms of collective coordinates is quadratic in \mathcal{M}^A 's. Since we need 8 exact zero modes and we have only 4 currents, it follows that each of these \mathcal{M}^A 's must be replaced by the exact zero modes. Therefore it suffices to solve eq.(5.4) in the presence of exact zero modes. In this case using eq.(2.14) one finds $\Phi^+ = 0$. This is due to the fact that $\mathcal{M}^A \mathcal{M}_A = 0$ for exact zero modes and the operator L is a positive definite operator. As a result χ_α is zero and eq.(5.6) solves eq.(5.4).

One can now, using eqs.(5.6),(2.16), compute the fermionic part of the $O(8)$ current for the 7-brane saddle point under consideration. The result turns out to be identical to the bosonic part of the current, namely the right hand side of (5.3). Thus the total $O(8)$ current is proportional to (5.3). It is also easy to check that the current is conserved ($\partial_\mu J_\mu = 0$).

The 8 exact gaugino zero modes can be grouped into an 8 component chiral spinor of 8

dimensional Euclidean space as

$$S = \begin{pmatrix} \psi_{\alpha 1} \\ \rho \xi_1^{\dot{\alpha}} \\ \psi_{\alpha 2} \\ \rho \xi_2^{\dot{\alpha}} \end{pmatrix} \quad (5.8)$$

Furthermore we will use the 8 dimensional (16×16) Γ matrices introduced in (4.4). Note that γ_μ and γ_5 act on $SO(4)_I$ indices α and $\dot{\alpha}$, and that, from the two sets of sigma matrices used, the first one acts on the $SU(2)_R$ index A, B while the second one acts on the two chiralities of 8-dimensional Euclidean space. Moreover the charge conjugation matrix C , as defined in section 4, raises and lowers the indices $\alpha, \dot{\alpha}$ and A . The current J_μ^a now becomes:

$$J_\mu^a(x) = \frac{1}{N^{1/4}g} \sum_{m,n=1}^5 S^T C (1 + \Gamma_9) \Gamma^{mn} S \mathcal{K} T^a \mathcal{K} \partial_{[m} G_{n]\mu}(X, \rho; x) \quad (5.9)$$

where G is the propagator of the gauge potentials on the AdS_5 space with $z = (X, \rho)$ being identified with a point in AdS_5 bulk [6, 25]

$$G_{n\mu}(X, \rho; x) = \frac{3}{\pi^2} \frac{\rho^2}{\rho^2 + (X - x)^2} J_{n\mu}(z - x) \quad (5.10)$$

where $J_{nm}(z) = \delta_{mn} - \frac{2z_m z_n}{z^2}$. Inserting the four $O(8)$ currents and integrating over the 8 exact gaugino zero modes produces the t_8 tensor and we get the result:

$$\begin{aligned} < \prod_{i=1}^4 J_{\mu_i}^{a_i}(x_i) >_{YM} = \int d^4 X \frac{d\rho}{\rho^5} d^3 \Omega \prod_{i=1}^4 < A_{\mu_i}^{a_i}(x_i) F_{m_i, n_i}^{b_i}(X, \rho) >_{AdS_5} \\ & \times t_8^{m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} Z_{b_1 b_2 b_3 b_4}^{(k)} \end{aligned} \quad (5.11)$$

where

$$Z_{b_1 b_2 b_3 b_4}^{(k)} = < \prod_{i=1}^4 \mathcal{K} T^{b_i} \mathcal{K} >_{D-inst} \quad (5.12)$$

Using either the computation of the D-string instantons in type I via the Matrix String approach or by using the heterotic world-sheet instanton result, which will be discussed in the next section, we conclude that

$$\begin{aligned} Z_{b_1 b_2 b_3 b_4}^{(k)} &= e^{2\pi i k \tau} \epsilon^{i_1 \dots i_8} J_{i_1 i_2}^{b_1} J_{i_3 i_4}^{b_2} J_{i_5 i_6}^{b_3} J_{i_7 i_8}^{b_4} \sum_{\ell|k} \frac{1}{\ell} \quad \text{for } k \text{ odd} \\ Z_{b_1 b_2 b_3 b_4}^{(k)} &= e^{2\pi i k \tau} [\text{tr}(J^{(b_1} J^{b_2} J^{b_3} J^{b_4)}) C_k^{F^4} \\ & \quad + \text{tr}(J^{(b_1} J^{b_2}) \text{tr}(J^{b_3} J^{b_4)}) C_k^{F^2 F^2}] \quad \text{for } k \text{ even} \end{aligned} \quad (5.13)$$

where

$$C_k^{F^4} = \begin{cases} 4 \sum_{\ell|k} \frac{1}{\ell} & \text{for } k = 4m - 2 \\ 4 \sum_{\ell|k} \frac{1}{\ell} - 4 \sum_{\ell|(k/2)} \frac{1}{\ell} & \text{for } k = 4m \end{cases}$$

and

$$C_k^{F^2 F^2} = \begin{cases} 2 \sum_{\ell|k} \frac{1}{\ell} & \text{for } k = 4m - 2 \\ 2 \sum_{\ell|k} \frac{1}{\ell} - \sum_{\ell|(k/2)} \frac{1}{\ell} & \text{for } k = 4m \end{cases}$$

In (5.13) J 's are the $O(8)$ generators in the vector representation, $\tau = a + i/g_{str}$ with a being the type I' 0-form RR potential and in the second equation on the right hand side b_i 's are totally symmetrized. Note that the k odd term above gets contribution only from the $O(8)$ odd parity invariant. This could be seen directly at the level of D-instanton action. Indeed the $O(8) \times O(k)$ fermions \mathcal{K} couple only to the $O(k)$ gauge fields b^+ in the D-instanton action. Integrating the \mathcal{K} therefore gives $(\det b^+)^8$ in the vector representation of $O(k)$. For odd k , b^+ has always at least one zero mode for generic values of b^+ . Thus $(\det b^+)^8$ has at least 8 zeroes, which are soaked by the 8 insertions of \mathcal{K} appearing in eq.(5.12) arising from the $O(8)$ current insertions. Soaking these zero modes brings about the 8-rank ϵ tensor in eq.(5.13) for odd k .

So far in the above discussion, the S^3 part of the measure has played no role apart from giving an overall volume factor. This is because we were considering correlators of the $O(8)$ currents which couple on the AdS_5 side to the lowest Kaluza-Klein modes of the $O(8)$ gauge fields on S^3 . There are two types of higher Kaluza-Klein modes one can consider: that of the closed string states such as dilaton, graviton etc. and that of the 7-brane world volume fields namely $O(8)$ gauge fields. For the closed string fields Kaluza-Klein modes will be on S^5/Z_2 while for the 7-brane fields these modes will be on S^3 and therefore one would expect the integral to be on the 7-brane world volume namely $AdS_5 \times S^3$. In particular the correlators involving Yang-Mills operators that couple to the Kaluza-Klein modes coming from the 7-brane fields, will get contribution only from 7-brane saddle point. These correlators can be calculated exactly as in the case of $N = 4$ theory discussed in [2] and one can show that they probe the S^3 part of the space-time.

5.2 Correlators related to bulk R^4 terms

We will now consider a correlator which will get contribution from the saddle point with $b^\pm \neq 0$ that probes the full $AdS_5 \times S^5/Z_2$. It is clear that such correlators must involve operators that do not carry $O(8)$ quantum numbers. Such operators can therefore be obtained by projection from the $\mathcal{N} = 4$ operators. In the $\mathcal{N} = 4$ context the operators $O^{ab} = \text{tr} \varphi^a \varphi^b$ in the (20) of $SO(6)$ (i.e. traceless symmetric tensor of 2 $SO(6)$ vectors, with a, b here labelling the $SO(6)$ vectors) carry dimension 2, and they couple via AdS/CFT correspondence to a certain Kaluza-Klein mode J of the scale factor h_α^α and the 4-form potential $C_{\alpha\beta\gamma\delta}$ on S^5 . The combination J of these two fields given by

$$J(x) = \int_{S^5} (h_\alpha^\alpha + \epsilon^{\alpha\beta\gamma\delta\rho} C_{\alpha\beta\gamma\delta} D_\rho) Y^{(2)} \quad (5.14)$$

where $Y^{(2)}$ is the second spherical harmonic on S^5 . In other words the relevant expansions of h_α^α and $C_{\alpha\beta\gamma\delta}$ are

$$h_\alpha^\alpha = J(x)Y^{(2)}, \quad C_{\alpha\beta\gamma\delta} = J(x)\epsilon_{\alpha\beta\gamma\delta\rho}D^\rho Y^{(2)} \quad (5.15)$$

The projection to $\mathcal{N} = 2$ splits (20) of $SO(6)$ into $(3, 3)_0$, $(1, 1)_{\pm 2}$ and $(1, 1)_0$ of $SU(2)_A \times SU(2)_Y \times U(1)$, the representations $(2, 2)_{\pm 1}$ being projected out. On the Yang-Mills side these operators are respectively $tr\phi^{AY}\phi^{BX}$ (where A, B and X, Y are symmetrized), $tr(\phi^\pm)^2$ and $tr\phi^+\phi^-$. On the AdS side these operators couple to field J mentioned above, where the spherical harmonics $Y^{(2)}$ is restricted to even ones under the Z_2 projection.

The simplest correlator is the one involving four operators $O_{(3,3)} = tr\phi^{AY}\phi^{BX}$ with A, B and X, Y symmetrized. In the large- N limit, we must replace the fields by their classical solution with the result:

$$O_{(3,3)} = \frac{1}{g^2} tr\bar{U}(\mathcal{M}^A f \mathcal{M}^{TY} + \mathcal{M}^Y f \mathcal{M}^{TA}) U \bar{U} \mathcal{M}^B f \mathcal{M}^{TX} U \quad (5.16)$$

The correlator then would involve 8 \mathcal{M}^A 's which will soak the 8 exact zero modes, leaving behind 8 \mathcal{M}^Y 's. This correlator gets contribution from both the saddle points described in the previous section:

1) When all the instantons sit at the same point in the 7-brane world volume (i.e. $b^\pm = 0$). In this case using the fact that ν^Y appear in the action in the combination $\nu^A \nu^Y b_{AY} + \nu^Y \nu_Y b^+$ and the fact that $b^+ = 0$ in this saddle point, we conclude that all the 8 \mathcal{M}^Y 's give rise \mathcal{M}'^Y 's and ζ^Y 's. The scaling of $(\mathcal{M}')^Y$ and ζ^Y by $\sqrt{g}N^{-1/8}$ and the measure factor Ng^4 in eq.(4.2) gives rise to a term which goes as order one in g and N times an $O(2r)$ Matrix Theory correlator of 8 $O(2r)$ adjoint fermions Λ 's in the matrix theory integral. In the supergravity theory this should correspond to 7-brane $t_8 R^4$ and $t_8(R^2)^2$ terms together with the terms involving the 4-form potential. Such terms via T-duality and S-duality should be obtainable from the heterotic one loop computation involving gravitons and anti-symmetric tensor fields.

2) When r instantons sit at a common point in $AdS_5 \times S^5/Z_2$ (with the other r instantons sitting at the image point) there are 8 zero modes of \mathcal{M}^Y 's as well that are not lifted and the resulting matrix theory is that of $SU(r)$ theory of type IIB instantons. Thus the result should be the same as in the type IIB case where this correlator is expected to get contribution from $t_8 t_8 R^4$ (and terms involving 4-form potential). The measure factor \sqrt{N} can be understood as follows. The 10-dimensional integral of R^4 goes as $L^2 \sim \sqrt{g^2 N}$ with L being the radius of S^5 . The factors of g cancel due to the fact that the leading D-instanton contribution to R^4 goes as $1/g$ [3, 4, 5]. Note that in type I' theory, the 0-form RR field χ is not projected out and therefore the 10-dimensional type IIB D-instanton contributions would survive in type I' theory.

6 String theory computation of F^4 terms

In this section we discuss the F^4 computation for $SO(8)^4$ heterotic theory in 8 dimensions. The world sheet instantons of the heterotic theory are then mapped via S-duality to D-string instantons in type I theory, which in turn are mapped via two T-dualities to (-1) -brane D-instantons in type I' theory. In the present context of the near horizon limit of the 3-branes lying at an orientifold plane where 8 of the 7- branes live, we are interested in all the four F's in the same $SO(8)$ factor. There are three independent quartic invariants of $SO(8)$:

$$\text{tr } F^4, \quad (\text{tr } F^2)^2, \quad \epsilon^{i_1 \dots i_8} F_{i_1 i_2} F_{i_3 i_4} F_{i_5 i_6} F_{i_7 i_8} \quad (6.1)$$

Here tr denotes trace in the vector representation of $SO(8)$ and ϵ is the 8-rank anti-symmetric tensor in the vector representation, $F_{ij} \equiv F_a (J^a)_{ij}$ with J^a being the $SO(8)$ generators in the vector representation. Note that the first two invariants are $O(8)$ parity even while the third one is $O(8)$ parity odd due to the ϵ tensor. The amplitudes involving the first two invariants, namely $\text{tr } F^4$ and $(\text{tr } F^2)^2$, have already been computed in [17, 18], with the result

$$\begin{aligned} \Delta^{F^2 F^2} &= 4[\log |\eta(4T)|^4 - 2 \log |\eta(2T)|^2] - 4 \log |T_2 U_2| \eta(U)|^4, \\ \Delta^{F^4} &= \frac{-1}{2} [\log |\eta(4T)|^4 - \log |\eta(2T)|^2] \end{aligned} \quad (6.2)$$

Here T is the complexified Kahler modulus of the torus $T = B_{12} + iR_1 R_2$. The coefficient of $e^{2\pi i n T}$ in the above denotes the contribution of n world sheet instantons to the amplitude. Note that there are only even number of instantons contributing to these two $SO(8)$ invariants. We shall now compute the last invariant, namely the $O(8)$ parity odd invariant. We will see that only odd instantons contribute to this term.

$SO(8)^4$ theory is best seen as a $Z_2 \times Z_2$ orbifold of heterotic $SO(32)$ theory compactified on a torus of radii $2R_1$ and $2R_2$. The orbifold group is generated by two elements g and h where g is half shift along first direction together with the Wilson line $(sp, sp, 0, 0)$ in the decomposition of $SO(32)$ in terms of $SO(8)^4$ and h is the half shift along the second direction together with the Wilson line $(sp, 0, sp, 0)$. Here $sp = (\frac{1}{2})^4$ denotes the highest weight of the spinor representation of $SO(8)$. Including all possible twists along σ and t directions of the world-sheet torus, we find 16 sectors which split into 5 modular orbits: one containing the completely untwisted sector, 3 orbits containing 3 sectors each and one orbit containing 6 twisted sectors. Denoting by (a, b) the twisted sectors with twist a along σ and b along t directions, we can represent these orbits as $(1, 1)$ for the completely untwisted sector, $(1, g)$, $(1, h)$ and $(1, gh)$ for the three orbits containing 3 sectors each and finally (g, h) for the orbit containing 6 sectors. The first 4 orbits contribute to the $O(8)$ even parity invariants and they have been already studied in [17, 18]. The last orbit represented by (g, h) contributes to the $O(8)$ odd parity term. The instanton numbers of course are governed by the windings along σ and t directions. Let us denote by (n_1, n_2) the windings of (X_1, X_2) along σ direction and (m_1, m_2) that

along t direction. World sheet instanton action is $(2R_1)(2R_2)|n_1m_2 - n_2m_1|$. Due to shifts associated with g and h the windings in these representatives of the 5 modular orbits are:

$$\begin{aligned}
(1, 1) & : (n_1, n_2; m_1, m_2) \\
(1, g) & : (n_1, n_2, m_1 + \frac{1}{2}, m_2) \\
(1, h) & : (n_1, n_2; m_1, m_2 + \frac{1}{2}) \\
(1, gh) & : (n_1, n_2; m_1 + \frac{1}{2}, m_2 + \frac{1}{2}) \\
(g, h) & : (n_1 + \frac{1}{2}, n_2; m_1, m_2 + \frac{1}{2})
\end{aligned} \tag{6.3}$$

Instanton number being invariant under the modular group, it is clear that in the first 4 orbits the instanton action is even number times $R_1.R_2$ while in the last orbit it is odd number times $R_1.R_2$.

Let us consider the $SO(32)$ fermion partition function. In the (g, h) sector, we have a twist along σ direction by $(sp, sp, 0, 0)$ and along t by $(sp, 0, sp, 0)$. This means that the characteristics of theta functions shift (in groups of four) by $(1/2, 1/2)$, $(1/2, 0)$, $(0, 1/2)$ and $(0, 0)$ respectively. Now in the original $SO(32)$ theory one has the sum over all spin structures for all the 32 fermions simultaneously. The shifts in the characteristics mentioned above imply that the partition function now becomes:

$$\begin{aligned}
\theta_3^{16} + \theta_4^{16} + \theta_2^{16} \pm \theta_1^{16} \rightarrow & \theta_1^4 \theta_2^4 \theta_3^4 \theta_4^4 + \theta_2^4 \theta_1^4 \theta_3^4 \theta_4^4 + \\
& \theta_4^4 \theta_3^4 \theta_1^4 \theta_2^4 + \theta_3^4 \theta_4^4 \theta_2^4 \theta_1^4
\end{aligned} \tag{6.4}$$

This partition function is of course zero because of the appearance of θ_1^4 in each term on the right hand side. However since F^4 vertex operators contain 4 Kac-Moody currents, they can in principle give non-zero answer. Indeed if we take for instance the four Kac-Moody currents to be in four different Cartan directions of a given $SO(8)$, then one of the terms in the right hand side will contribute since the effect of introducing these currents is to take derivatives of the appropriate $\theta(\tau, z)$ with respect to z at $z = 0$. The result therefore becomes:

$$\theta_1^4 \theta_2^4 \theta_3^4 \theta_4^4 = 2^8 \pi^4 \eta^{24} \tag{6.5}$$

Remarkably η^{24} cancels the $1/\eta^{24}$ coming from the partition function of the oscillator modes in the bosonic string sector (the right moving sector) of heterotic string. Left moving part of the heterotic string, i.e. the fermionic string sector, gives the usual t_8 tensor in the presence of 4 gauge field vertices as can be easily seen in the Green-Schwarz formalism. Indeed each of the F vertex contains two Green-Schwarz fermions that soak the 8 fermion zero modes in the light cone gauge and all the non-zero mode determinants cancel due to space-time supersymmetry. Thus the final result for the coefficient $Z(T, U)$ of the $O(8)$ odd parity term $\epsilon^{i_1 \dots i_8} F_{i_1 i_2} F_{i_3 i_4} F_{i_5 i_6} F_{i_7 i_8}$ is just given by

the 2-dimensional lattice sum of the windings.:

$$Z(T, U) = \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2^2} 4T_2 \sum_M e^{8\pi i |det M| T} e^{-\frac{4\pi T_2}{\tau_2 U_2} |(1, U)M \begin{pmatrix} \tau_1 \\ -1 \end{pmatrix}|^2} \quad (6.6)$$

where

$$M = \begin{pmatrix} n_1 + \frac{1}{2} & m_1 \\ n_2 & m_2 + \frac{1}{2} \end{pmatrix}, \quad (6.7)$$

with n_i and m_i being integers. U is the complex structure of the torus and \mathcal{M} is the fundamental domain of the Γ_2 subgroup of $SL(2, Z)$ defined as the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with b and c even integers. Note that \mathcal{M} has 6 copies of the fundamental domain of $SL(2, Z)$ owing to the fact that the orbit of (g, h) consists of 6 twisted sectors. One can now use suitable elements of Γ_2 to set $n_2 = 0$ and the result is the unfolding of the domain \mathcal{M} to the full upper half plane. The integration over the upper half plane yields the result:

$$Z(T, U) = 2(\log |\eta(T)|^2)_{odd} \quad (6.8)$$

where the subscript *odd* indicates projection onto odd powers of $e^{2\pi i T}$. Notice that there is no U dependence since in this sector there is no degenerate orbit.

Equations (6.2) and (6.8) give the complete F^4 terms in the $SO(8)^4$ heterotic theory in 8-dimensions. In type I also one can do an analogous calculation involving D-string instantons wrapped on the torus following the methods of [16, 15]. The basic idea here is that in the infrared limit, the theory of N D-strings collapses into a symmetric product of N copies of heterotic string (in the static gauge). The instanton contribution is obtained by considering the one loop amplitude involving 4 F vertices with σ and t being identified with X_1 and X_2 respectively, as is dictated by the static gauge. We will not give the details here but a trivial extension of the results of [16, 15] after including the Wilson lines in g and h yields the same result as the heterotic string one loop result. Finally in the type I' theory which is obtained by 2 T-dualities on type I theory, the coupling F^4 should be the same as in type I, however a direct computation of D-instanton contributions in type I' theory is an interesting open question.

7 Conclusions

In this paper we studied the multi-instanton effects in $\mathcal{N} = 2 Sp(N)$ Yang-Mills theory that appears in the 3-brane world volume in the type I' theory (2 T-dual of type I). We showed that the ADHM construction of the multi-instantons give rise to collective coordinates which can be interpreted as the fields living in the (-1) -brane instantons in the presence of 3-branes in type I'. The saddle points in the large N-limit describes (-1) brane instantons sitting at various points in the 7-brane world volume ($AdS_5 \times S^3$) or in the 10-dim bulk ($AdS_5 \times S^5/Z_2$) that appears as the near horizon limit of the 3-brane

solutions. We further discussed the nature of the exact saddle points in the large- N limit and argued that there are two distinct classes of exact saddle points, one where all the instantons sit at the same point in the 7-brane world volume and the other when all the instantons sit at a common point in the 10-dim bulk. The latter is of course possible only for even number of instantons since there has to be an equal number of instantons at a given point and its image under the Z_2 action. While the 7-brane saddle point measure is order 1 in N , the bulk saddle point measure goes as \sqrt{N} .

In the 7-brane saddle point there are 8 exact fermion zero modes and the measure splits into an integral over $AdS_5 \times S^3$ times a matrix integral over the (-1) brane world volume theory ($O(k)$ gauge theory) that appears in type I' where the center of mass variables have been factored out. This matrix integral via 2 T-dualities is related to D-string instanton integrals in type I (which can be evaluated in the Matrix-String approach of [26]) and via a further S-duality is related to 1-loop world-sheet instantons in heterotic theory. We computed 4 $O(8)$ current correlator in the Yang-Mills theory which receives contribution only from the 7-brane saddle point as is to be expected from the fact the $O(8)$ gauge fields live only in the 7-brane world volume, and showed that the result is obtainable from F^4 terms in the 7-brane world volume via AdS/CFT correspondence. A novel feature we find is that also the odd instantons contribute to such correlators in the Yang-Mills side and to F^4 term on the string theory side where it appears as $O(8)$ odd parity quartic invariant. In particular this shows that BPS instantons break $O(8)$ to $SO(8)$ in contrast with the situation in 10-dim type I theory where $O(32)$ is broken to $SO(32)$ due to the non BPS Z_2 instanton [27].

We also discussed some Yang-Mills correlators that do not involve $O(8)$ quantum numbers. In this case only the even instantons contribute and both the saddle points are relevant. These correlators are given by the R^4 couplings (and their supersymmetric partners) on the 7-brane world volume and in the 10-dim. bulk respectively for the 2 saddle points:

1) The 7-brane R^4 coupling is of the form $t_8 R^4$ and $t_8 (R^2)^2$ that appears via T and S-duality at 1-loop in the heterotic theory. The (-1) -brane instanton contribution to such term is given by 1-loop world sheet instanton effects in the heterotic theory. These are the super invariants I_1 and I_3 in the notation of [28], which are related to anomaly cancelling CP odd term and therefore are expected to satisfy non-renormalization theorem. In the type I' side this result should be obtainable by computing 8 fermion correlator (these are the zero modes that are lifted and carry (-1) charge under $O(2)$ R-symmetry) in the $O(2k)$ matrix integral (where the center of mass has been factored out). It is an interesting open problem to work out this matrix integral and show that it correctly reproduces the tensor structure of the I_1 and I_3 invariants as well as the coefficients in the instanton expansion. However we expect this to be true due to the non-renormalization theorem associated with these invariants.

2) More interesting and perhaps somewhat unexpected is the contribution of the 10-dim

bulk saddle point. In this case besides the 8 exact supersymmetric and superconformal zero modes (that transform as $(2, 1)_{+1}$ under $SU(2)_A \times SU(2)_Y \times O(2)$) there are 8 more exact fermion zero modes that transform as $(1, 2)_{-1}$. The measure factorizes into an integral over the bulk $AdS_5 \times S^5/Z_2$ together with these 16 fermion zero modes times $SU(k)$ matrix integral (where $2k$ is the number of instantons) appearing in the type IIB context. This is exactly the situation for the k type IIB (-1) brane instantons and therefore should produce $t_8.t_8 R^4$ term (and its supersymmetric partners). This superinvariant (called I_0 in [28]) appears at the tree level and the one loop torus level in type I and type I' (since the computation is the same as in type IIB) as well as at the tree level in the heterotic theory. I_0 superinvariant is not expected to satisfy any non-renormalization theorem. Indeed it has been argued in [28] that the heterotic-type I duality demands that I_0 receives correction to all loops already in the 10-dimensional theory. In the type I' theory under consideration (i.e. IIB on $T^2/Z_2.(-1)^{F_L}.\Omega$) the (-1) -brane instantons of the IIB theory are not projected out. Therefore in the large volume limit of T^2 we expect the (-1) -brane instanton contributions to the I_0 in IIB theory in 10-dim to survive. This is exactly the term which contributes to the instanton effects in the corresponding Yang-Mills correlators arising from the bulk saddle point. In fact this saddle point is dominant and goes as \sqrt{N} in the large N-limit, due to the fact that integral is also over the 2-directions transverse to 7-brane and its volume goes as $\sqrt{g_{st}N}$ (the factor $\sqrt{g_{st}}$ cancels from the $1/\sqrt{g_{st}}$ that appears in the leading instanton contribution to R^4 [3, 4, 5]).

An interesting open question is what is the fate of the type I' I_0 invariant in type I and heterotic theory. By the duality relations one finds that the leading instanton term in type I' is mapped to

$$\begin{aligned} \frac{T_{I'}}{\sqrt{g_{I'}}} e^{-\frac{k}{g_{I'}}} \int d^8 x R_{I'}^4 &\rightarrow \frac{1}{\sqrt{T_I \cdot g_I}} e^{-\frac{k T_I}{g_I}} \int d^8 x R_I^4 \\ &\rightarrow \frac{g_h}{\sqrt{T_h}} e^{-k T_h} \int d^8 x R_h^4 \end{aligned} \quad (7.1)$$

where subscripts I' , I and h refer to the variables in the three theories, and T is the volume of the torus. Note that in the heterotic theory this expression does not make sense due to the odd power of g_H . This problem already appears at the 10-dimensional level, as discussed by [28] with a possible resolution being that both in type I and heterotic side this term should receive corrections to all orders.

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